Artificial Intelligence
CE-417, Group 1
Computer Eng. Department
Sharif University of Technology

Spring 2023

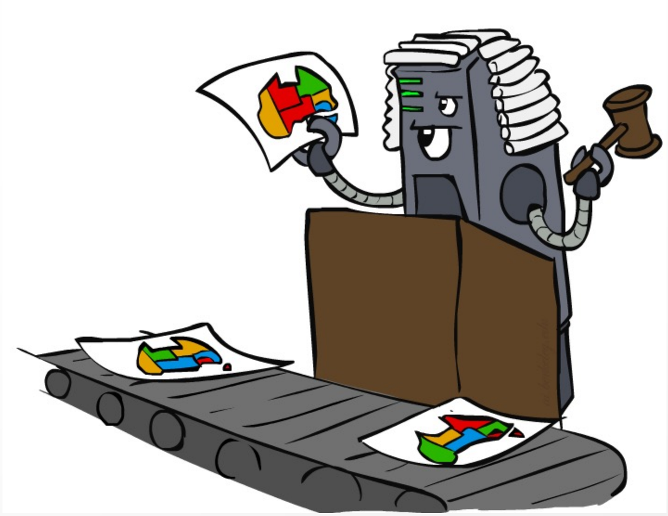
By Mohammad Hossein Rohban, Ph.D.

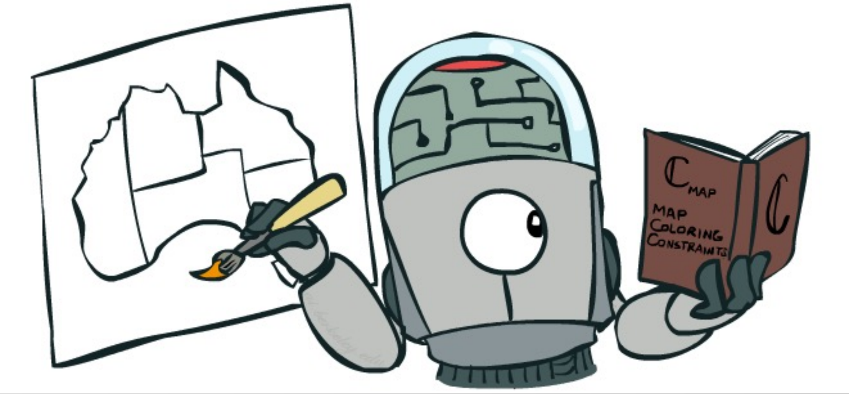
Courtesy: Most slides are adopted from CSE-573 (Washington U.), original slides for the textbook, and CS-188 (UC. Berkeley).

Constraint Satisfaction Problems

Constraint satisfaction problems (CSPs)

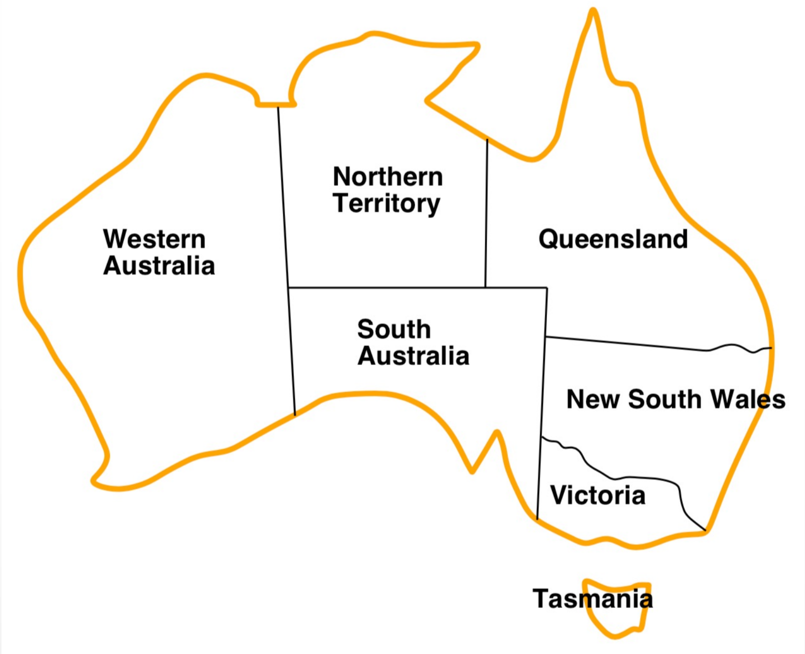
- Standard search problem:
  - state is a “black box”
  - any old data structure that supports goal test, evaluation, and successor
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variable
  - Allows useful general-purpose algorithms with more power than standard search algorithms





Example: Map-Coloring

CSO

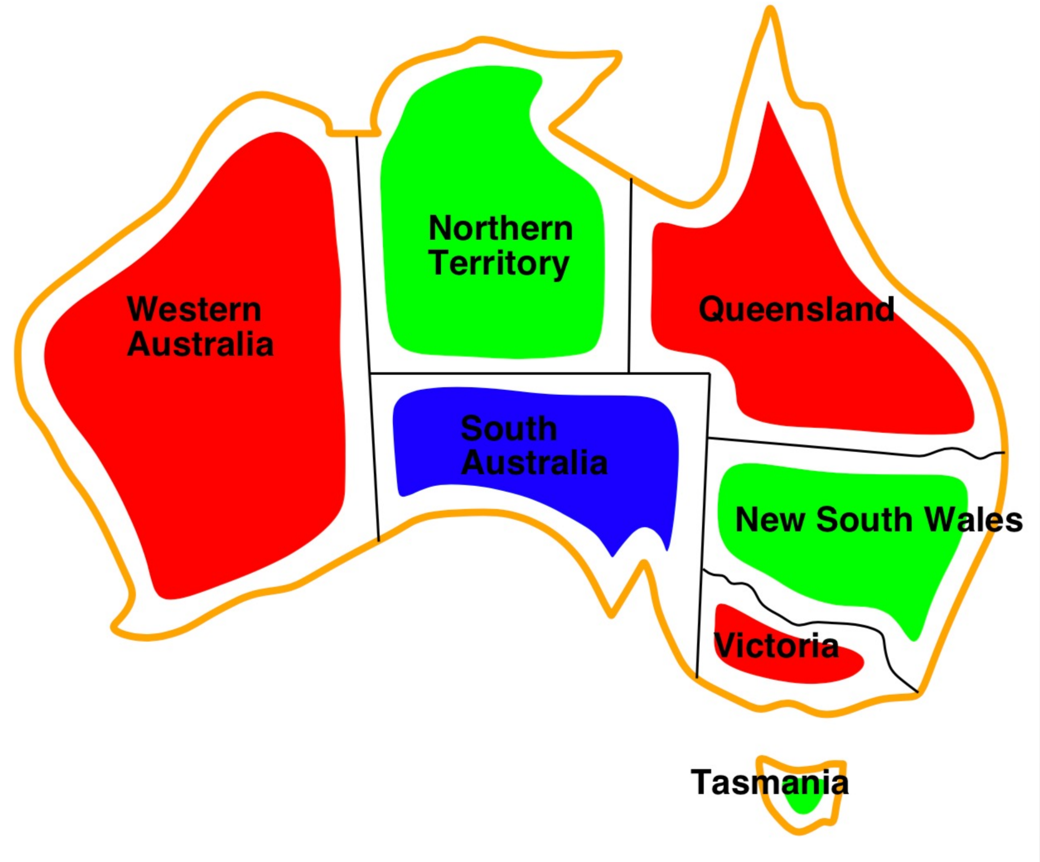


- Variables WA, NT, Q, NSW, V , SA, T
- Domains D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors

  e.g., WA $\neq$ NT (if the language allows this), or

(WA, NT) $\in$ {(red, green), (red, blue), (green, red), (green, blue), . . .}

Example: Map-Coloring (cont.)

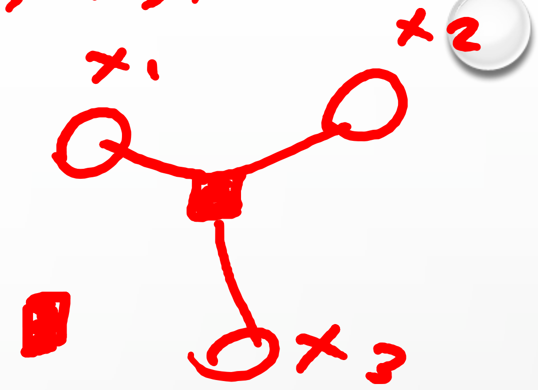


Solutions are assignments satisfying all constraints, e.g.,

{WA=red, NT =green, Q=red, NSW =green, V =red, SA=blue, T =green}

Constraint graph $C(x_i, x_j)$

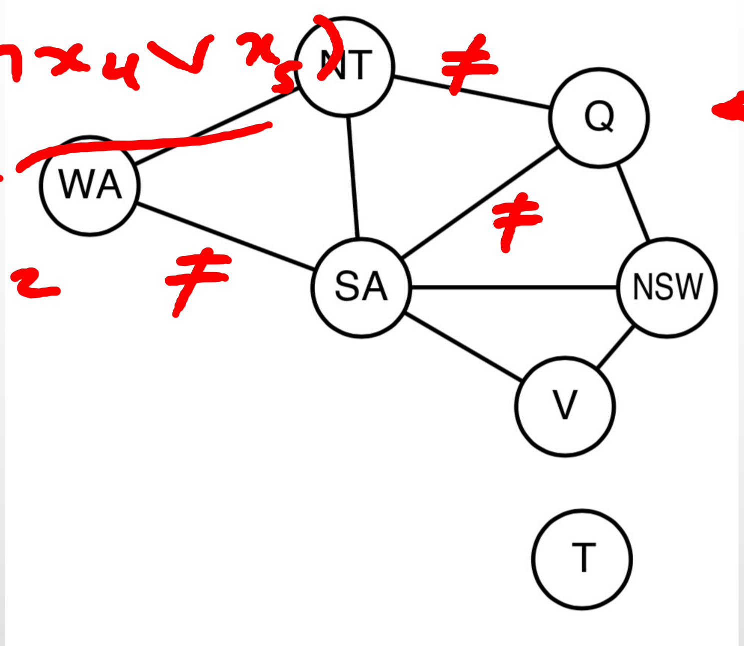
$x_1 \dots x_n$



- Binary CSP: each constraint relates at most two variables
- Constraint graph: nodes are variables, arcs show constraints

3-SAT

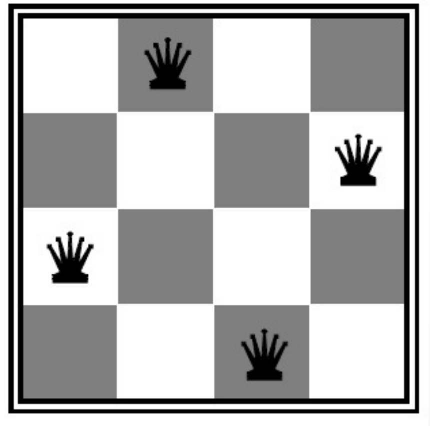
$$\underbrace{(x_3 \vee x_1 \vee x_2)}_{C_1} \wedge \underbrace{(x_1 \vee \neg x_4 \vee x_5)}_{C_2}$$

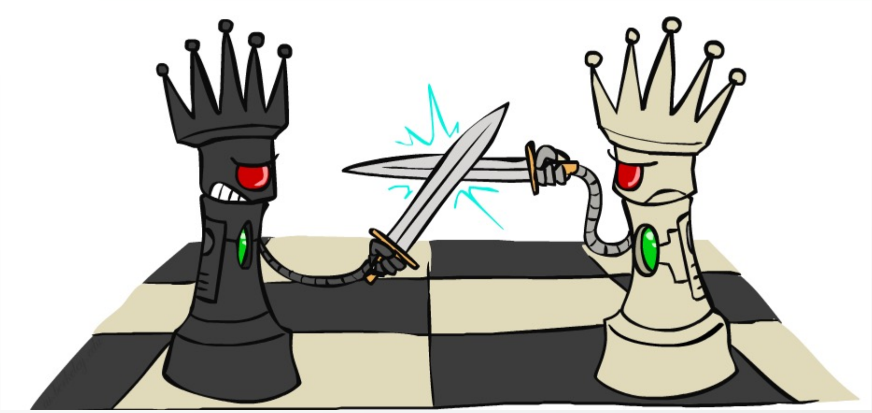


$$\left\{ \begin{array}{l} C(x_i, x_j) = T \Rightarrow \\ x_i \neq x_j \end{array} \right.$$

- General-purpose CSP algorithms use the graph structure to speed up search. e.g., Tasmania is an independent subproblem!

Example: n-queens





- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\underline{\{0, 1\}}$
  - Constraints

$$\forall i, j, k \ \ \overline{(X_{ij}, X_{ik})} \in \{(0,0),(0,1),(1,0)\}$$

$$\forall i, j, k \ \ (\underline{X_{ij}, X_{kj}}) \in \{(0,0),(0,1),(1,0)\}$$

$$\forall i, j, k \ \ (\underline{X_{ij}}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$

$$\forall i, j, k \ \ (\underline{X_{ij}, X}_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

$$\underline{\sum_{i,j} X_{ij} = N}$$

Example: Cryptarithmetic

- Variables:

  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

- Domains:

  $\{0,1,2,3,4,5,6,7,8,9\}$

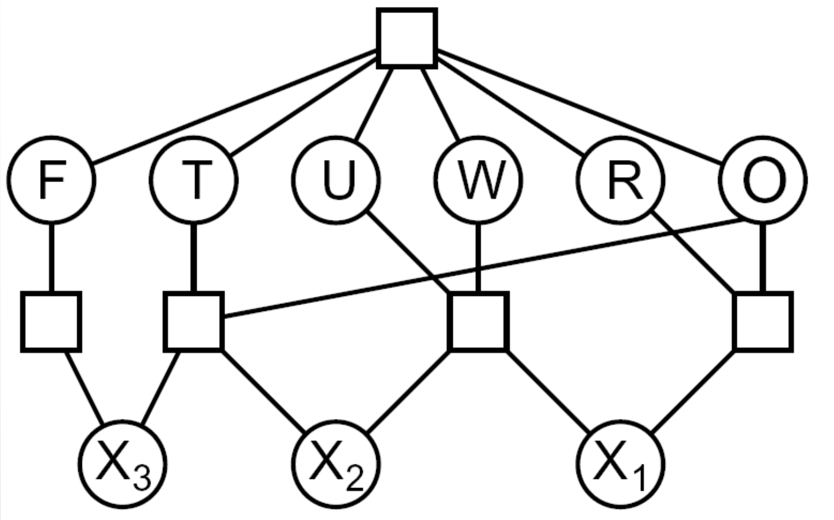
- Constraints:

  $\text{alldiff}(F,T,U,W,R,O)$

  $O+O=R+10\cdot X_1$

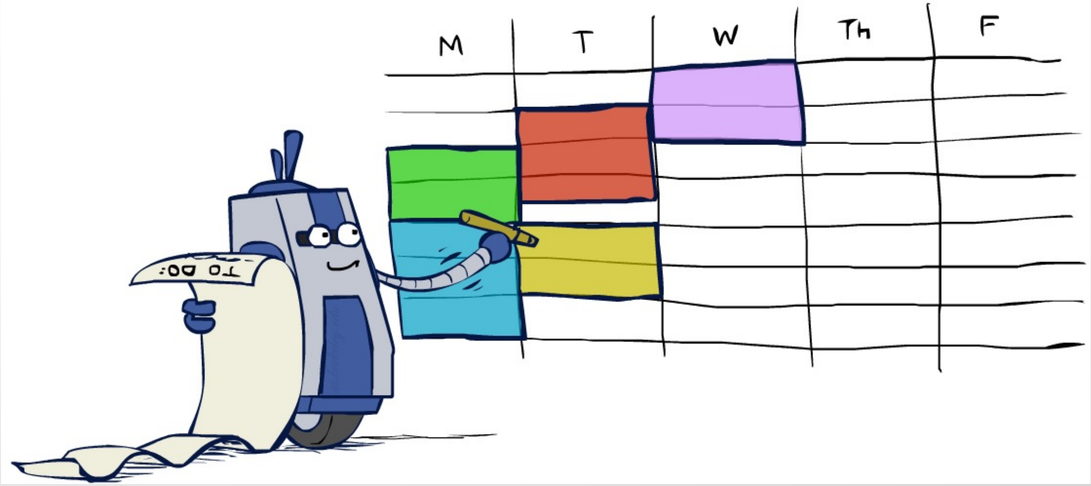
  $\cdots$

$$\begin{array}{r} \text{T W O} \\ +\ \text{T W O} \\ \hline \text{F O U R} \end{array}$$



Real-World CSPs

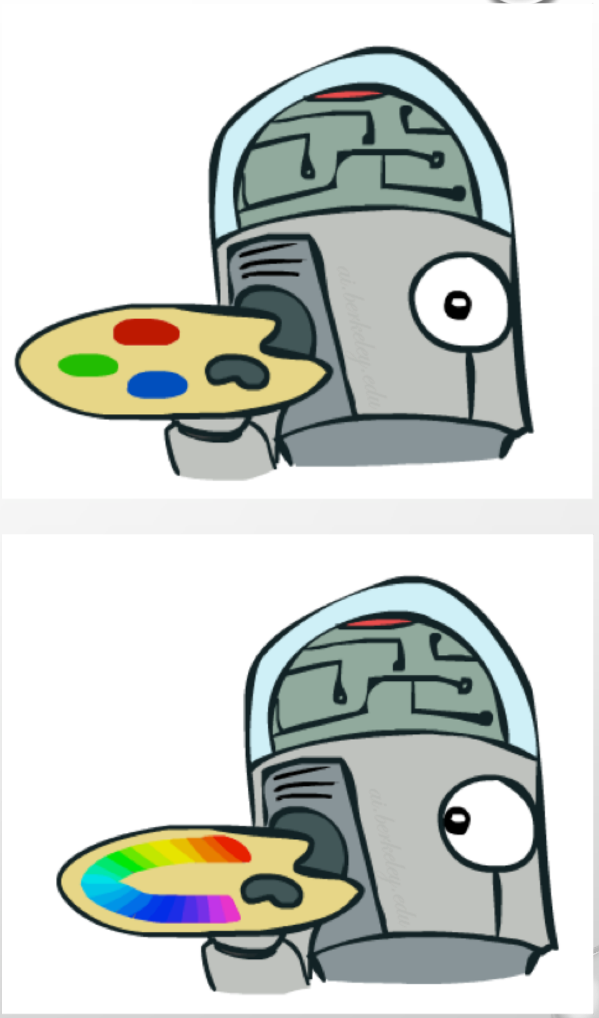
- Assignment problems: e.g., Who teaches what class
- Timetabling problems: e.g., Which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … Lots more!



- Many real-world problems involve real-valued variables…

Varieties of CSPs

- Discrete variables
  - finite domains; size $d \Rightarrow O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  - infinite domains (integers, strings, etc.)
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $\text{StartJob}_1 + 5 \leq \text{StartJob}_3$
    - linear constraints solvable, nonlinear undecidable
- Continuous variables
  - e.g., start/end times for Hubble Telescope observations
  - linear constraints solvable in poly time by LP methods



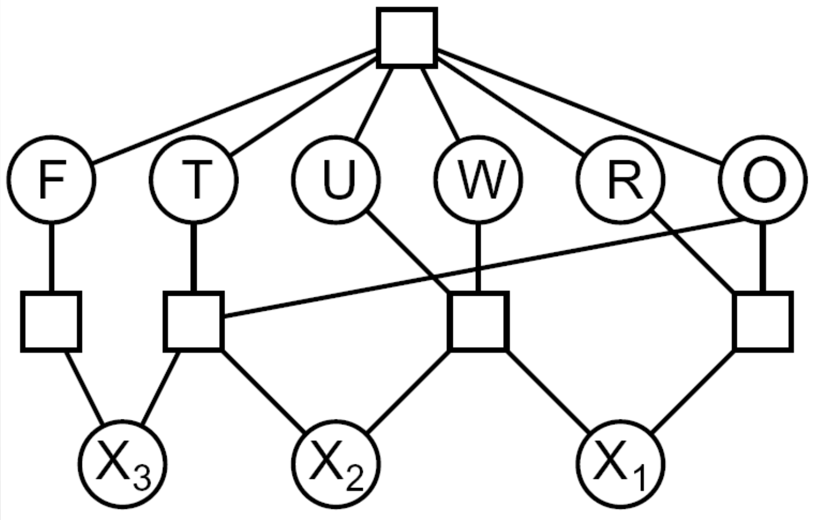
Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables, e.g., cryptarithmetic column constraints
- Preferences (soft constraints), e.g., *red* is better than *green* often representable by a cost for each variable assignment

  → constrained optimization problems

Converting n-ary CSP to a binary CSP

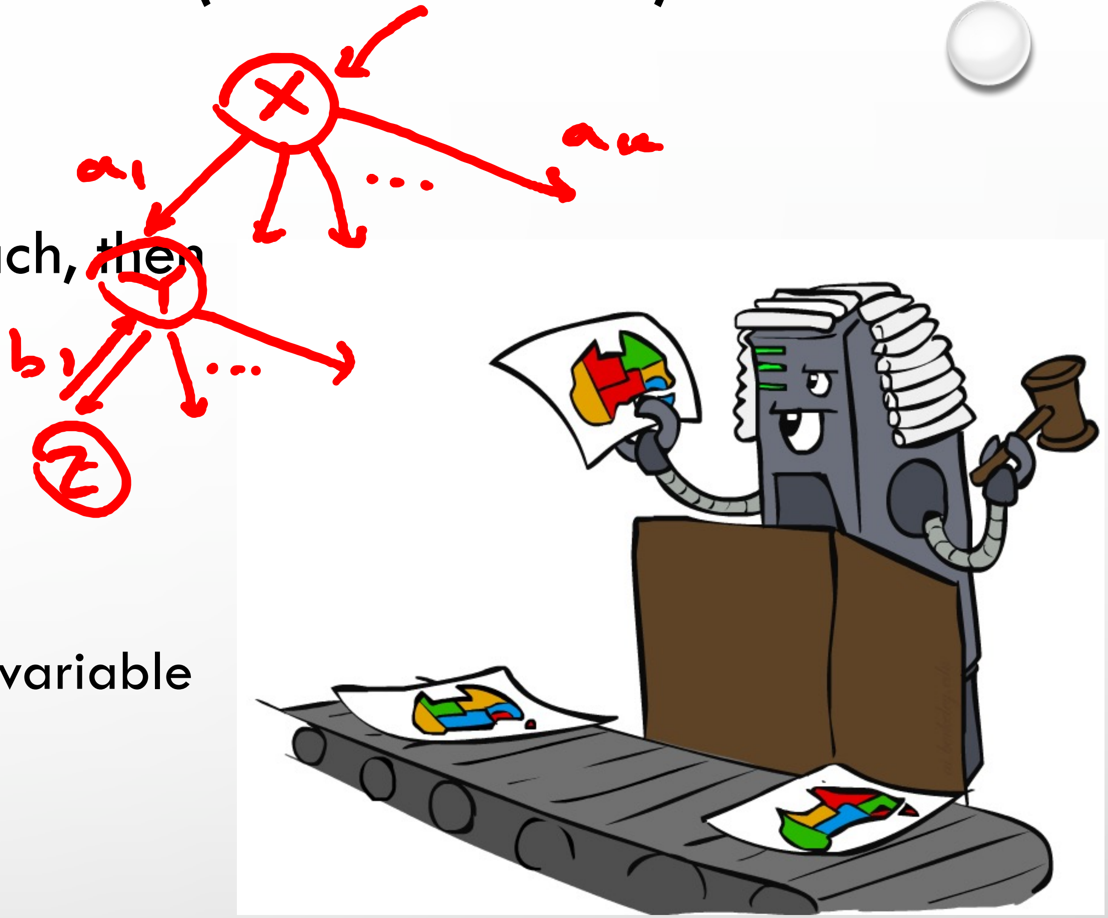
- Is this possible?



Standard search formulation (incremental)

- Let's start with the straightforward, dumb approach, then fix it.
- States are defined by the values assigned so far
  - Initial state: the empty assignment, { }
  - Successor function: assign a value to an unassigned variable that does not conflict with current assignment.

    ⇒ fail if no legal assignments (not fixable!)
  - Goal test: the current assignment is complete



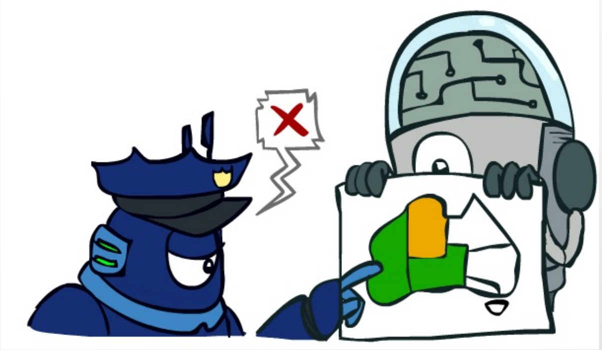
Standard search formulation (incremental) (cont.)

- This is the same for all CSPs!
- Every solution appears at depth n with n variables ⇒ use depth-first search
- Path is irrelevant, so can also use complete-state formulation.
- $b = (n - l)d$ at depth l, hence $n!\, d^n$ leaves!!!!

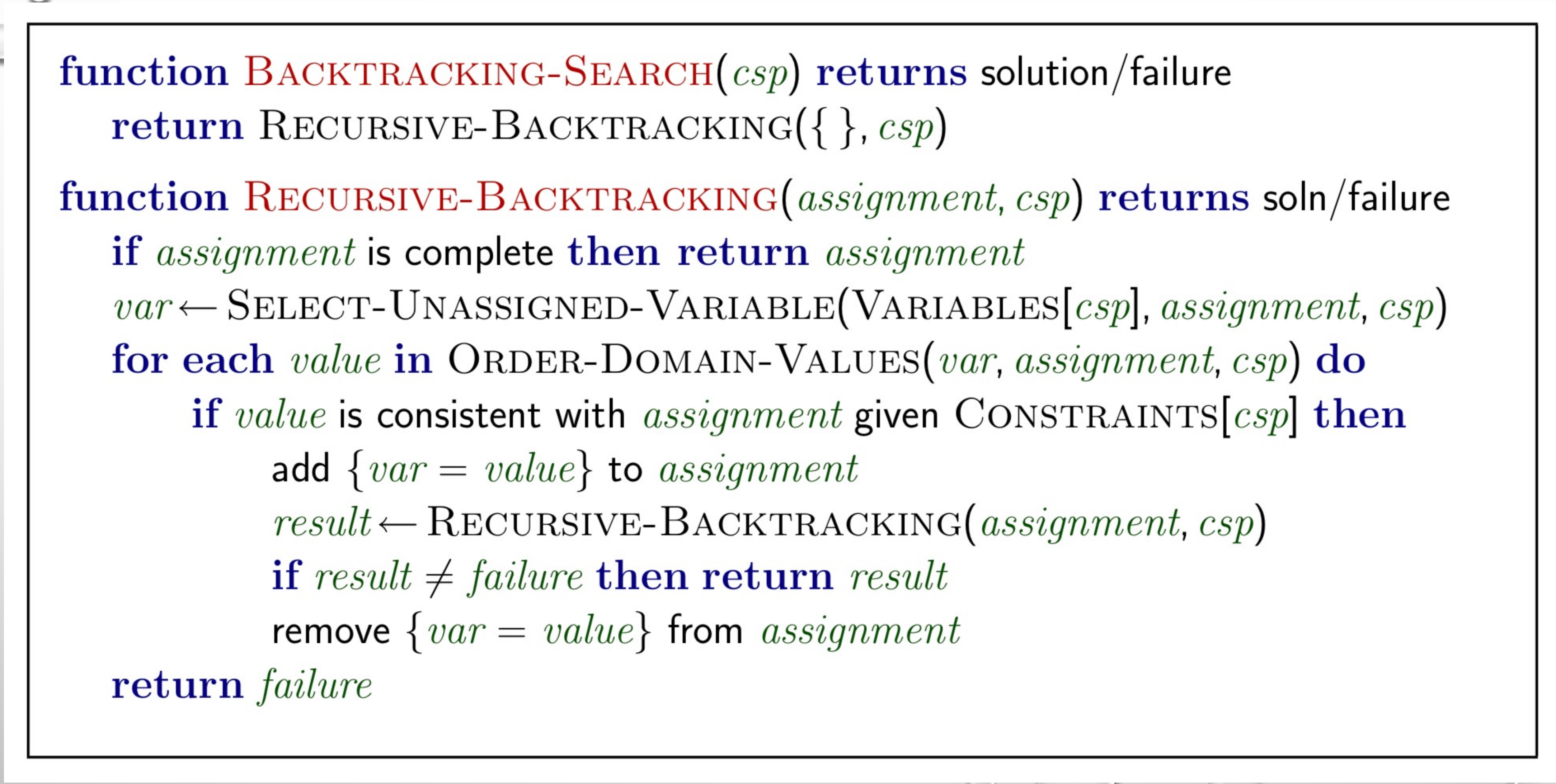
Backtracking search

- Variable assignments are commutative, i.e.,
  [WA=red then NT =green] same as [NT =green then WA=red]
- Only need to consider assignments to a single variable at each node

  ⇒ b=d and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search.
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25



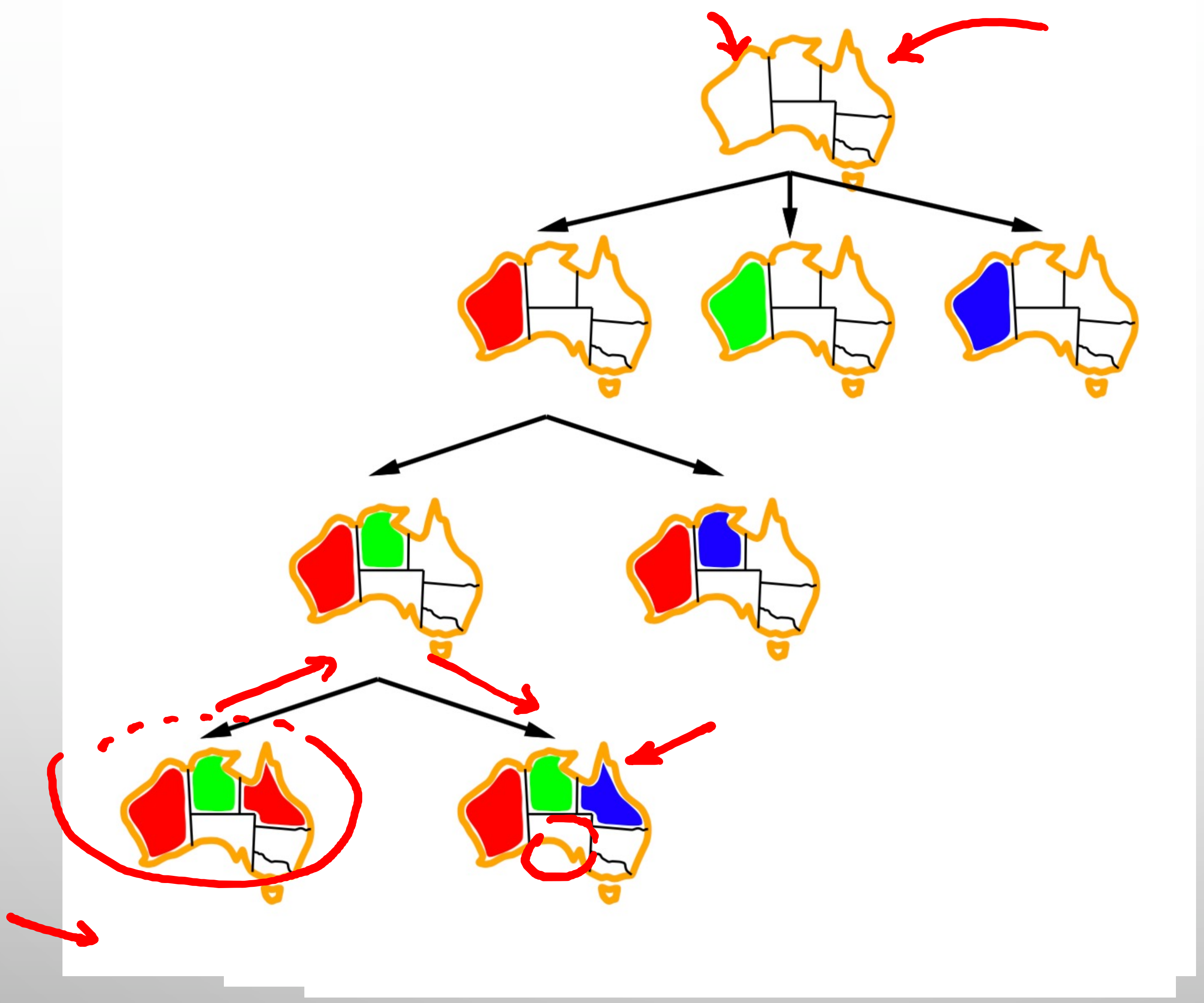
Backtracking search



```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Backtracking search (cont.)

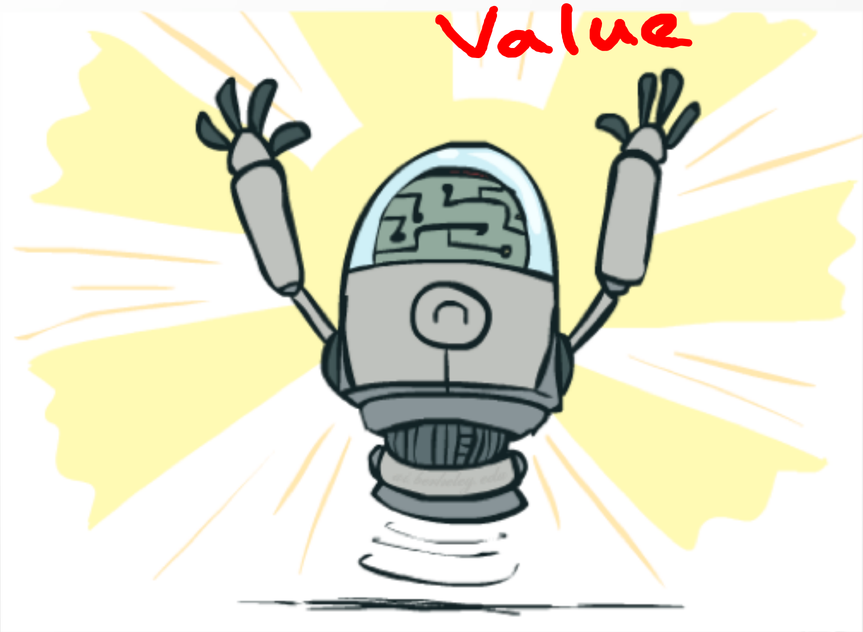


Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

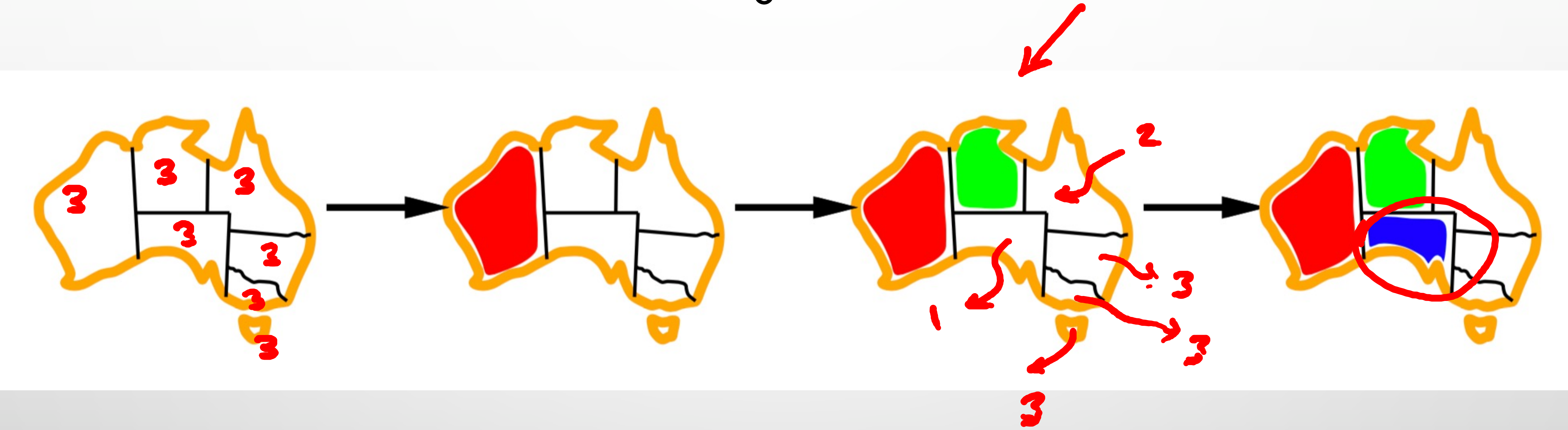
1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

minimum remaining Value



Minimum remaining values

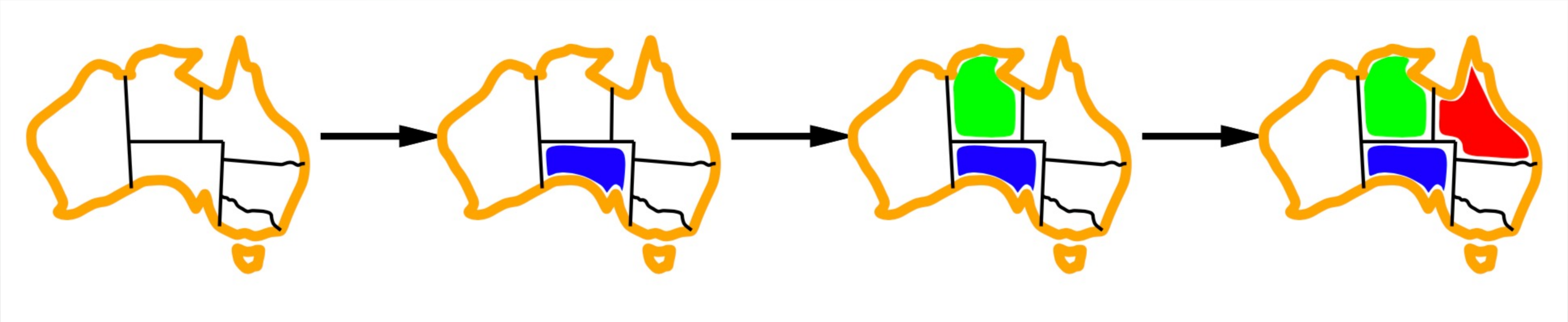
- Minimum remaining values (MRV):
  choose the variable with the fewest legal values



Degree heuristic

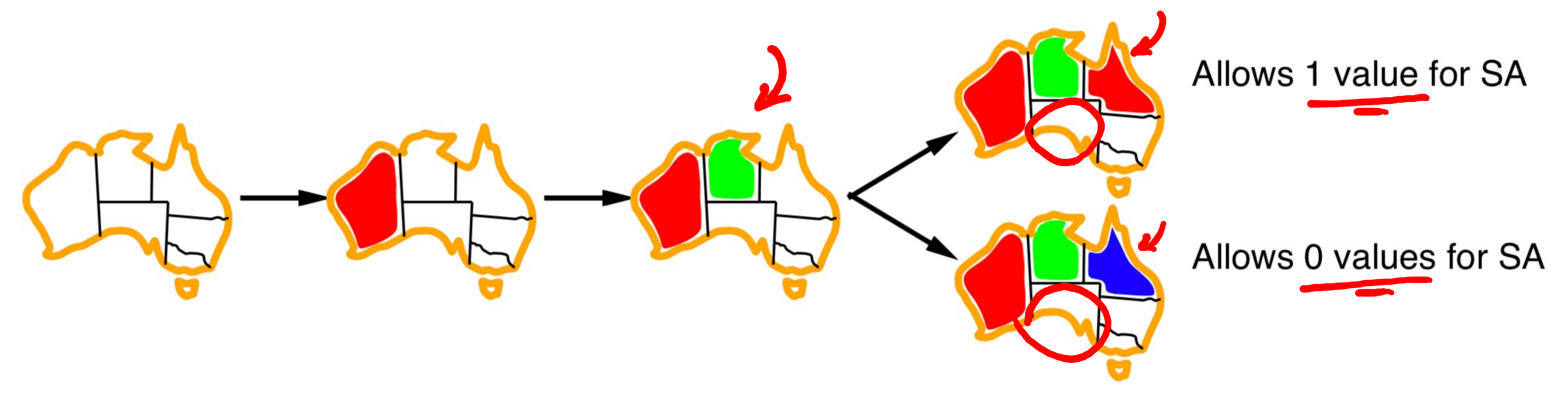
- Tie-breaker among MRV variables
- Degree heuristic:

  choose the variable with the most constraints on remaining variables



Least constraining **value**

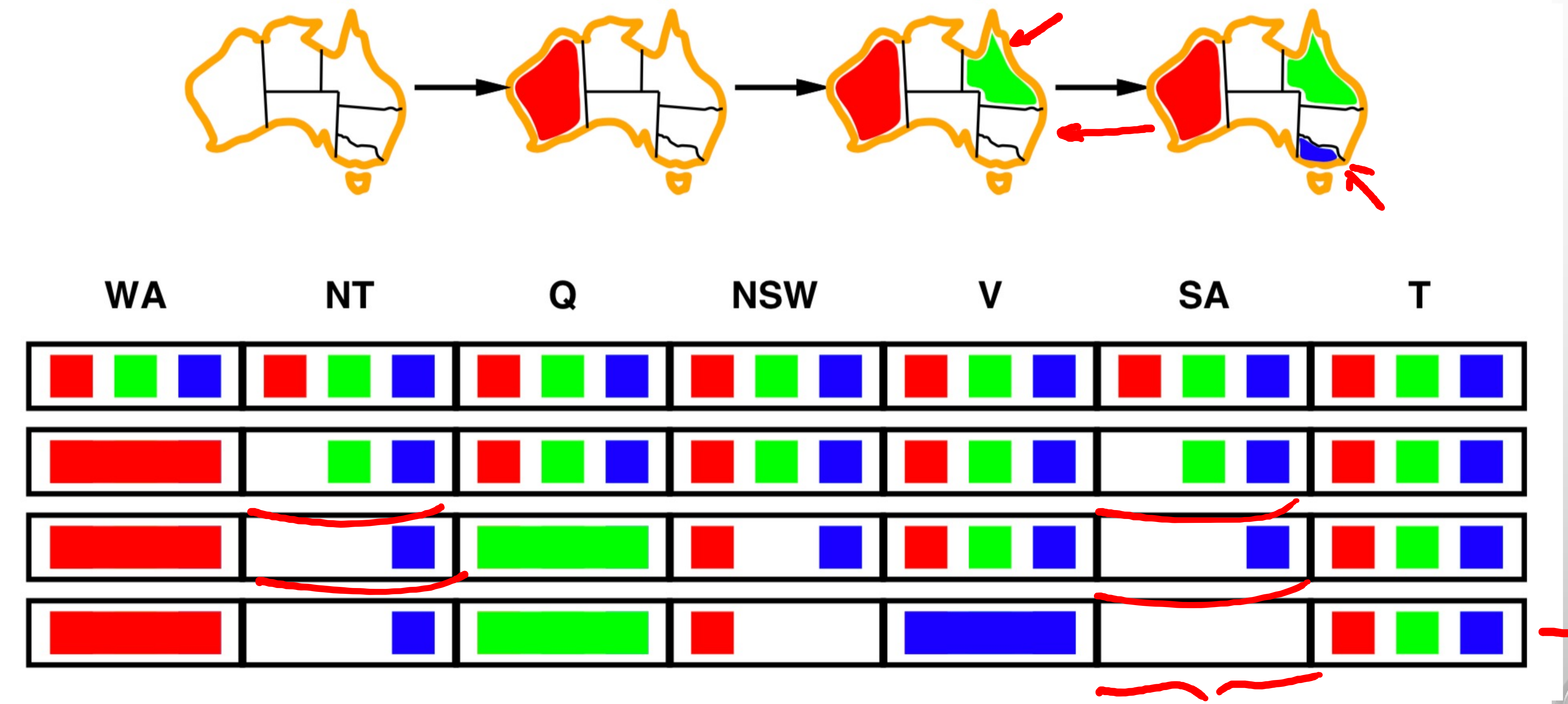
- Given a variable, choose the least constraining value: the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea: Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

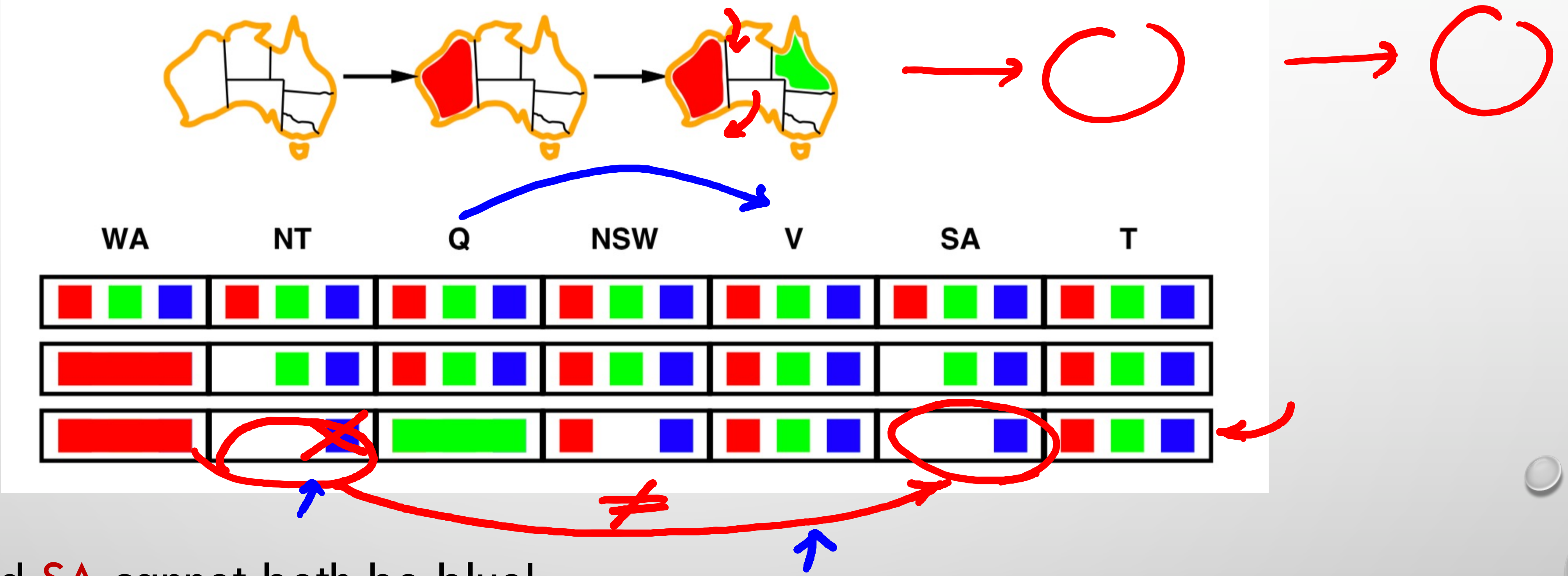


WA	NT	Q	NSW	V	SA	T
🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦
🟥	🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟩🟦	🟥🟩🟦
🟥	🟦	🟩	🟥🟦	🟥🟩🟦	🟦	🟥🟩🟦
🟥	🟦	🟩	🟥	🟦		🟥🟩🟦

Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

dead end



WA	NT	Q	NSW	V	SA	T
🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦
🟥	🟩🟦	🟥🟩🟦	🟥🟩🟦	🟥🟩🟦	🟩🟦	🟥🟩🟦
🟥	🟦	🟩	🟥🟦	🟥🟩🟦	🟦	🟥🟩🟦

≠

- NT and SA cannot both be blue!

Constraint propagation repeatedly enforces constraints locally

$\min xy + z$

x, y, z

...

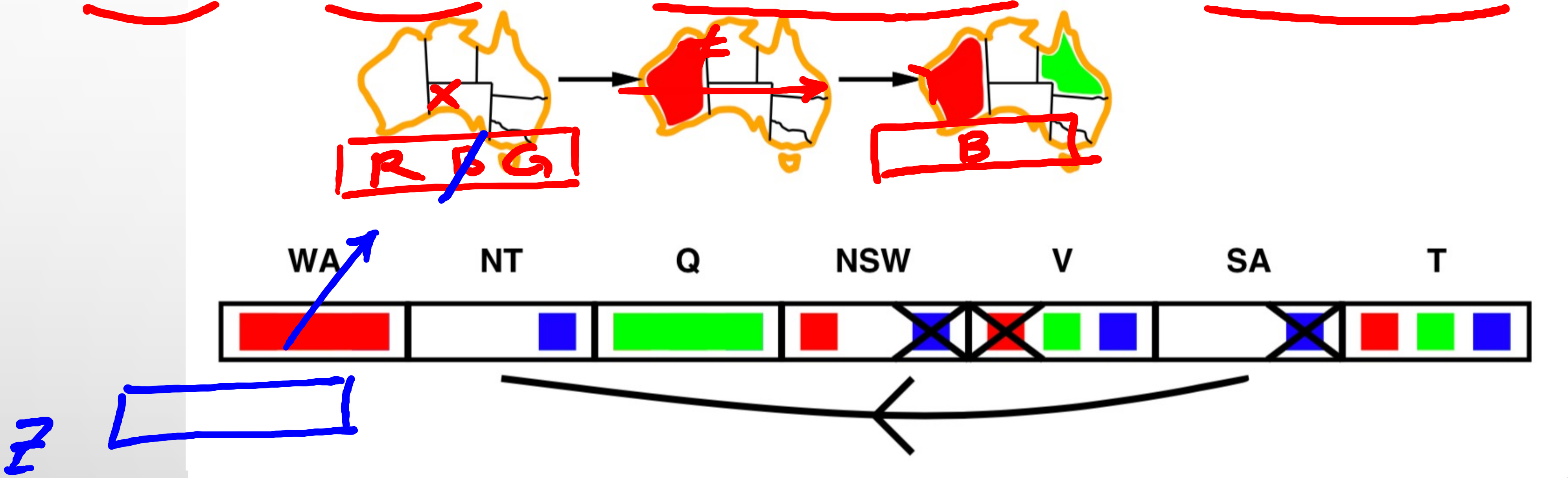
$z \geq 0$ → u^2

Constraint Graph

Arc consistency

$x + y = 5$

- Simplest form of propagation makes each arc consistent
- X → Y is consistent iff. for **every** value x of X there is **some** allowed y



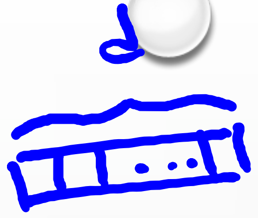
- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking. Can be run as a preprocessor or after each assignment

Arc consistency algorithm

```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty do
        $(X_i, X_j) \leftarrow$ REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES($X_i$, $X_j$) then
            for each $X_k$ in NEIGHBORS[$X_i$] do
                add $(X_k, X_i)$ to queue
```

```
function REMOVE-INCONSISTENT-VALUES( $X_i$, $X_j$) returns true iff succeeds
    removed $\leftarrow$ false
    for each $x$ in DOMAIN[$X_i$] do
        if no value $y$ in DOMAIN[$X_j$] allows $(x,y)$ to satisfy the constraint $X_i \leftrightarrow X_j$
            then delete $x$ from DOMAIN[$X_i$]; removed $\leftarrow$ true
    return removed
```



$X_i \rightarrow X_j$

added

would be ✓ to Q most $\underline{d}$ times

$O(d^2) \leftarrow$

$\cdot$

$O(n^2)$

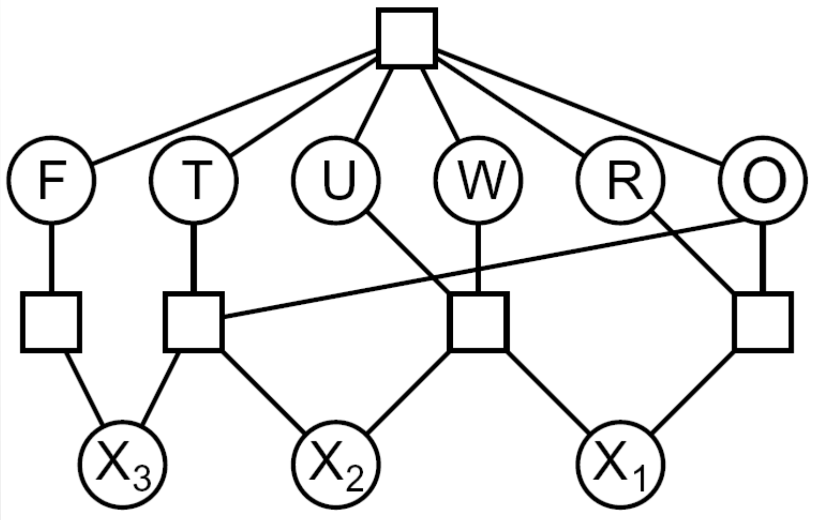
- $O(n^2d^3)$, can be reduced to $O(n^2d^2)$

n = # variables

d = Domain size (# values)

Arc consistency for n-ary CSP?

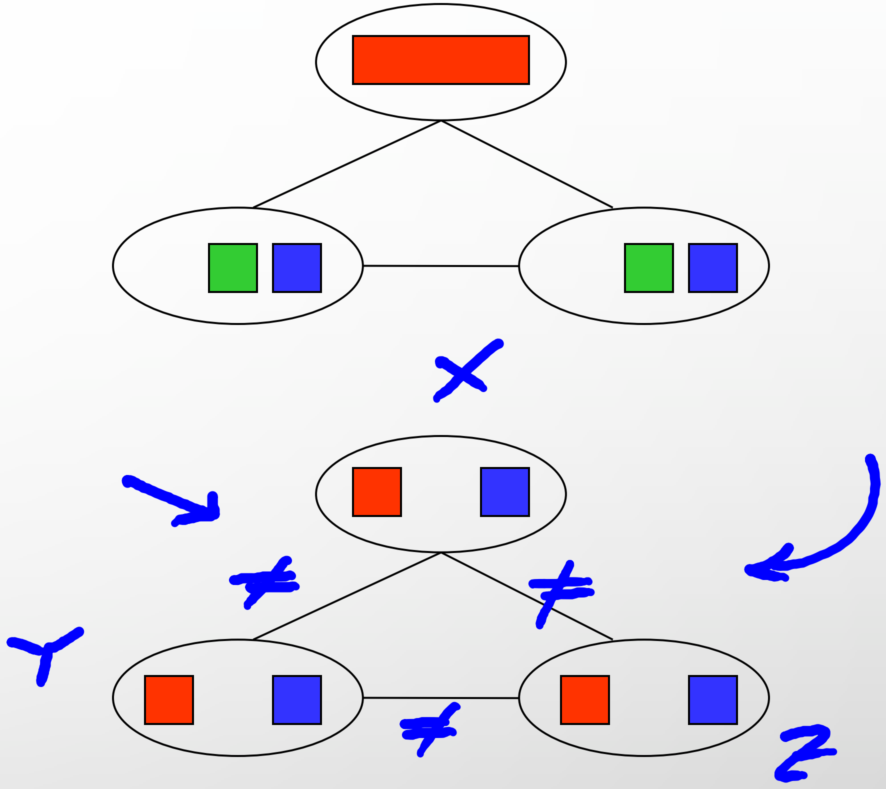
- How to generalize to the n-ary CSP case?



Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

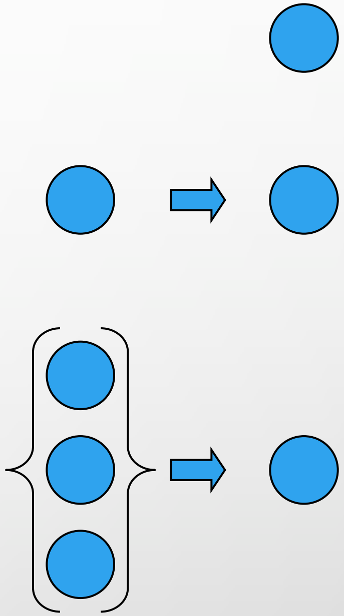
- Arc consistency still runs inside a backtracking search!



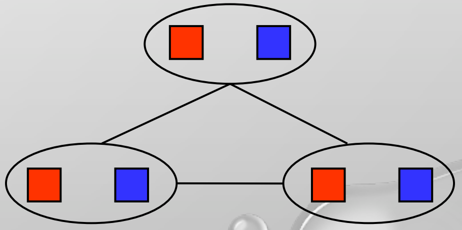
What went wrong here?

k-Consistency

- Increasing degrees of consistency
  - 1-consistency (node consistency): each single node's domain has a value which meets that node's unary constraints
  - 2-consistency (arc consistency): for each pair of nodes, any consistent assignment to one can be extended to the other
  - k-consistency: for each k nodes, any consistent assignment to k-1 can be extended to the kth node.



- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



Strong k-Consistency

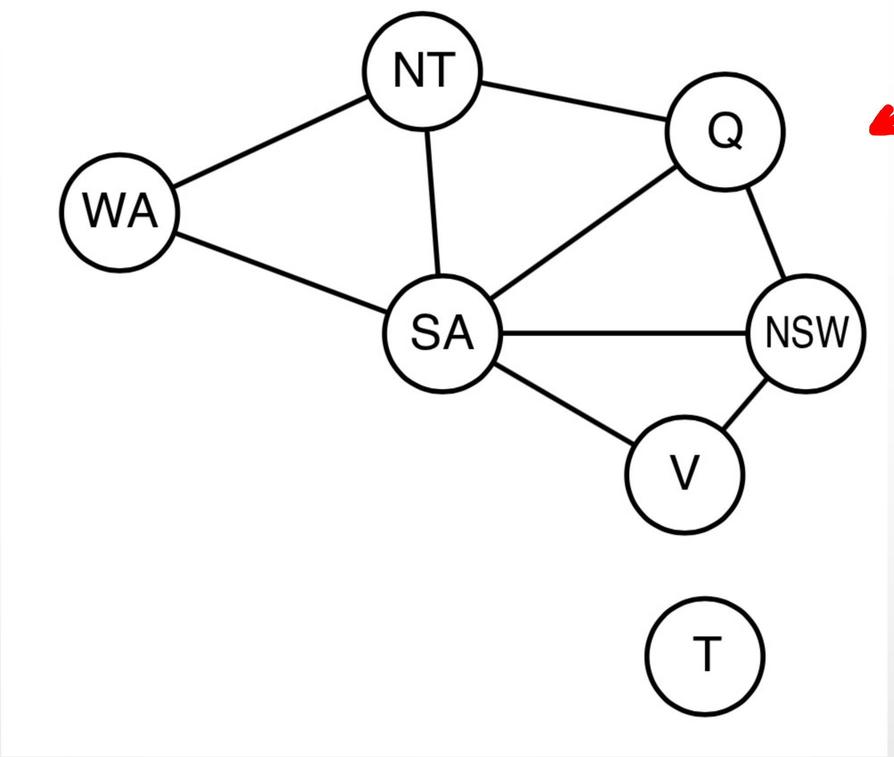
- Strong k-consistency: also k-1, k-2, … 1 consistent

- Claim: strong n-consistency means we can solve without backtracking!

- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …

- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Problem structure



- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph

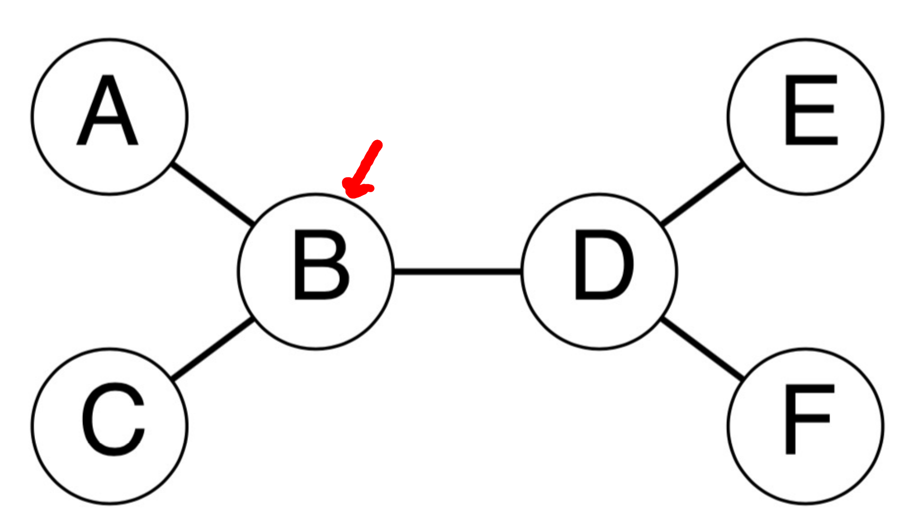
Problem structure (cont.)

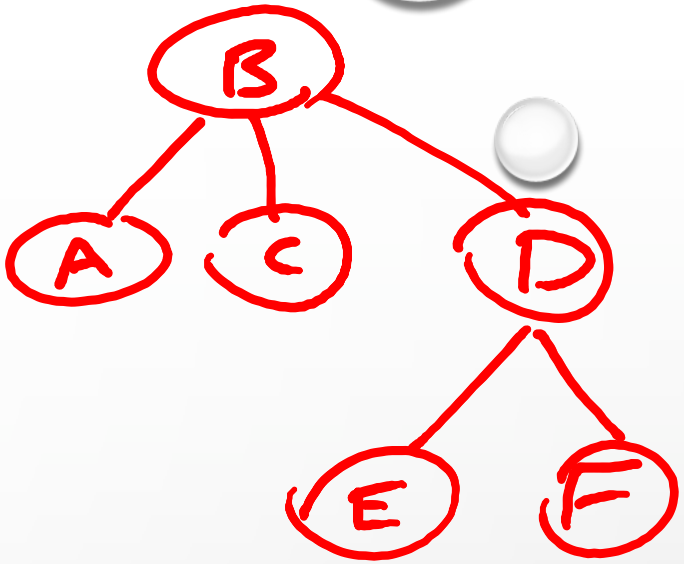
- Suppose each subproblem has c variables out of n total
- Worst-case solution cost is $n/c \cdot d^c$, linear in n
- E.g., $n=80, d=2, c=20$
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs

Topological Sort

B, A, C, D, F, E

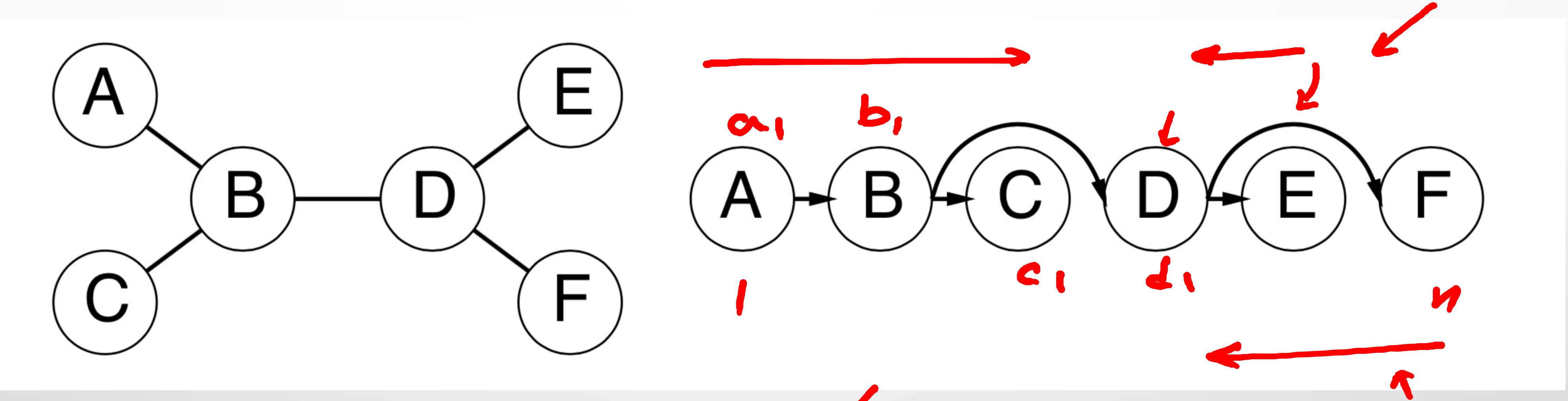




- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n.d^2)$ time.
- Compare to general CSPs, where worst-case time is $O(d^n)$.
- This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering

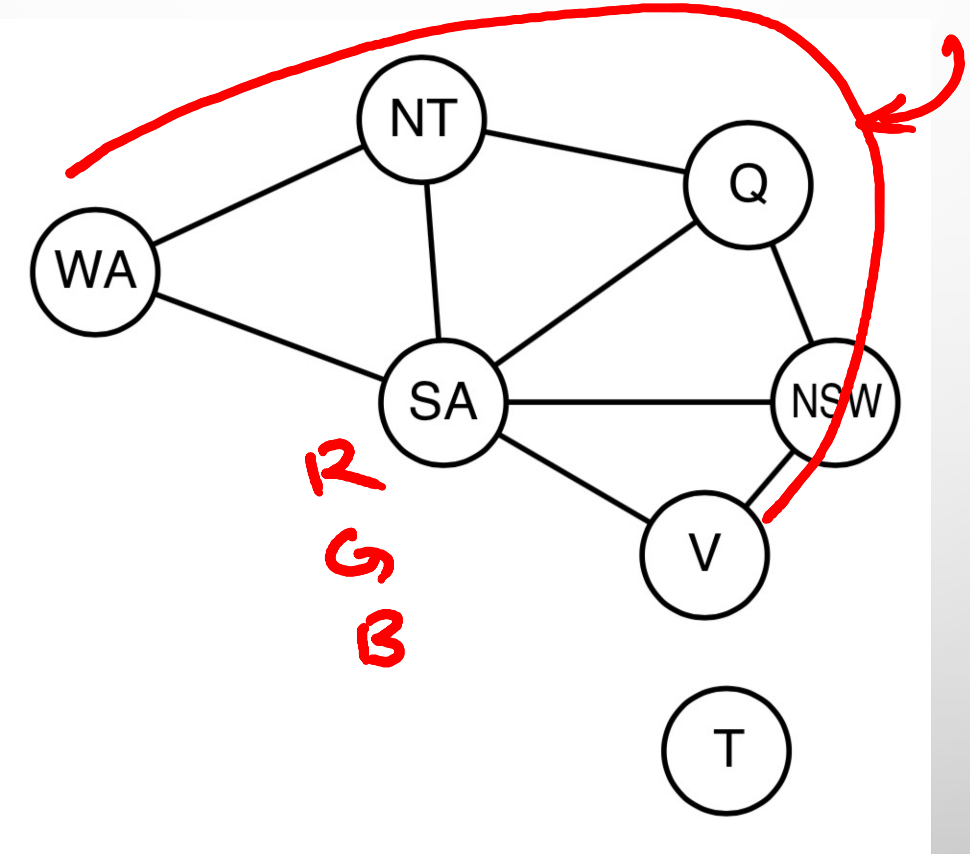


2. For j from n down to 2, apply RemoveInconsistent(Parent(X_j), X_j)
3. For j from 1 to n, assign X_j consistently with Parent(X_j).

Why doesn't this algorithm work with cycles in the constraint graph?

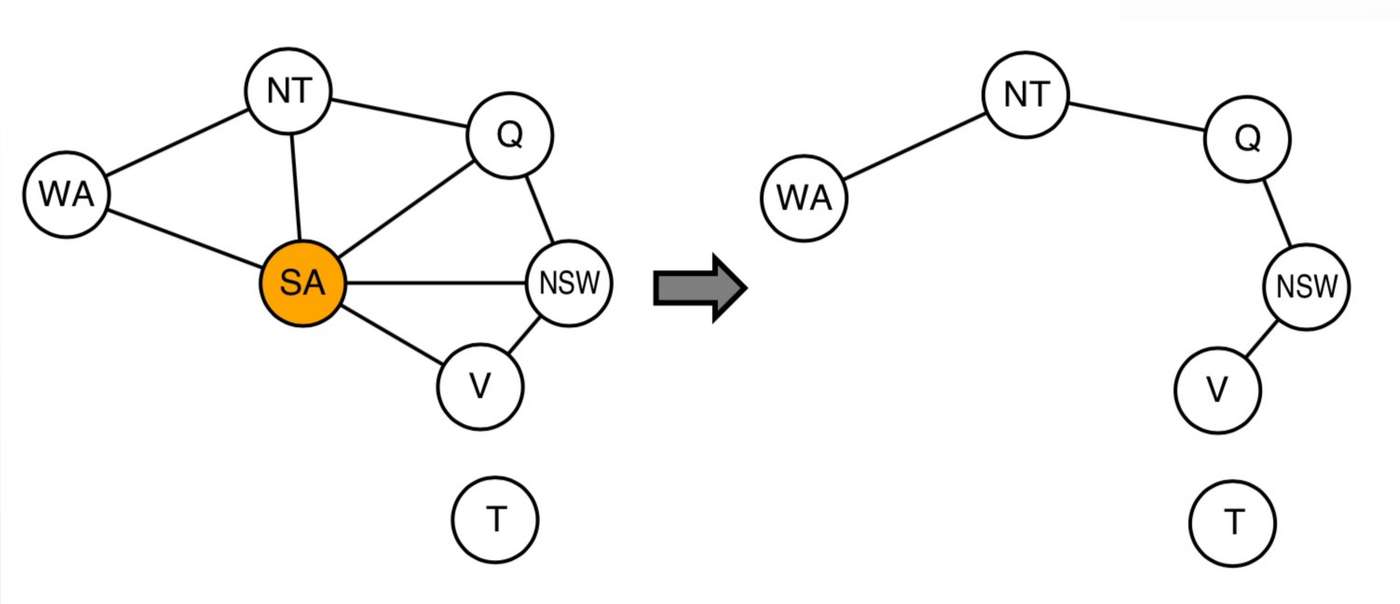
Nearly tree-structured CSPs

- How to solve the CSP corresponding to this constraint graph using tree structured CSP?



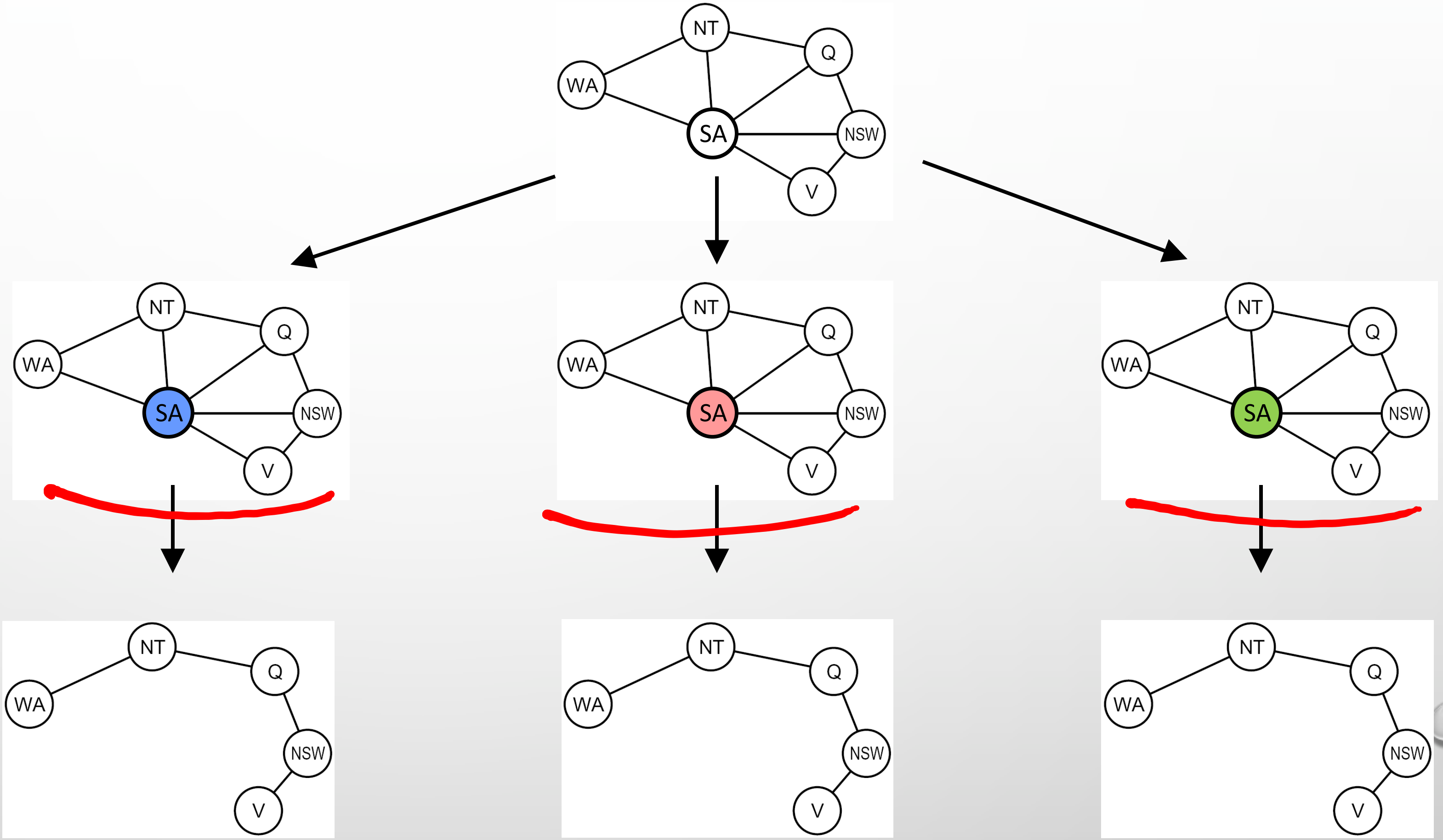
Nearly tree-structured CSPs (cont.)

- Conditioning: instantiate a variable, prune its neighbors' domains



- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c ⇒ runtime $O(d^c \cdot (n - c)d^2)$, very fast for small c.

Cutset Conditioning



Choose a cutset

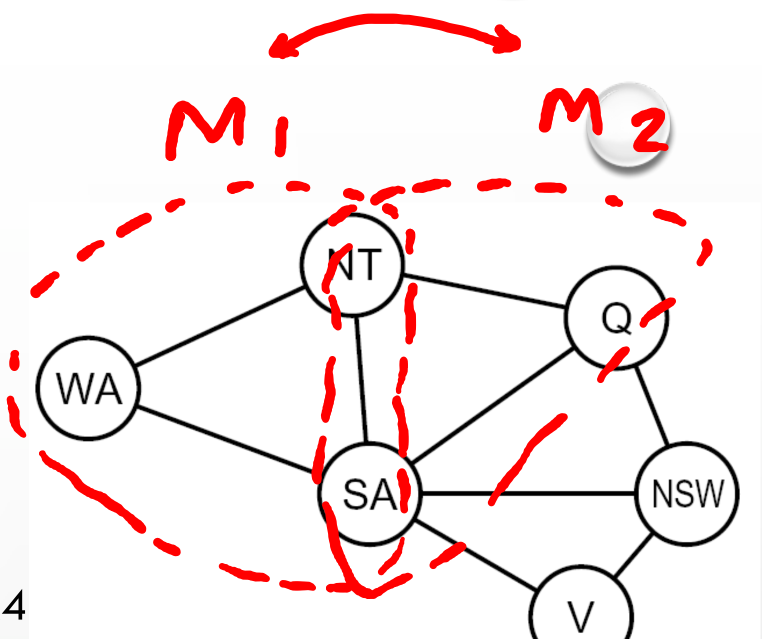
Instantiate the cutset (all possible ways)

Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)

Tree Decomposition

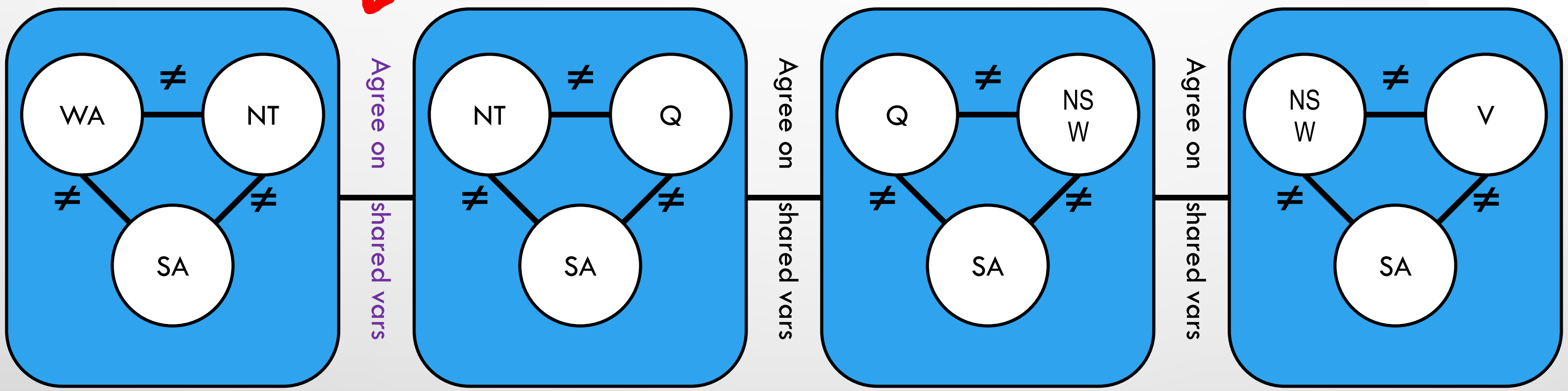
- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions



$O(d^3)$

2 2

M1 M2 M3 M4



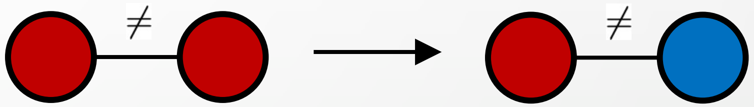
{(WA=r,SA=g,NT=b),
(WA=b,SA=r,NT=g),
…}

{(NT=r,SA=g,Q=b),
(NT=b,SA=g,Q=r),
…}

Agree: (M1,M2) ∈
{((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

Iterative algorithms for CSPs

- Hill-climbing, simulated annealing typically work with “complete” states, i.e., all variables assigned.
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - Operators: reassign variable values



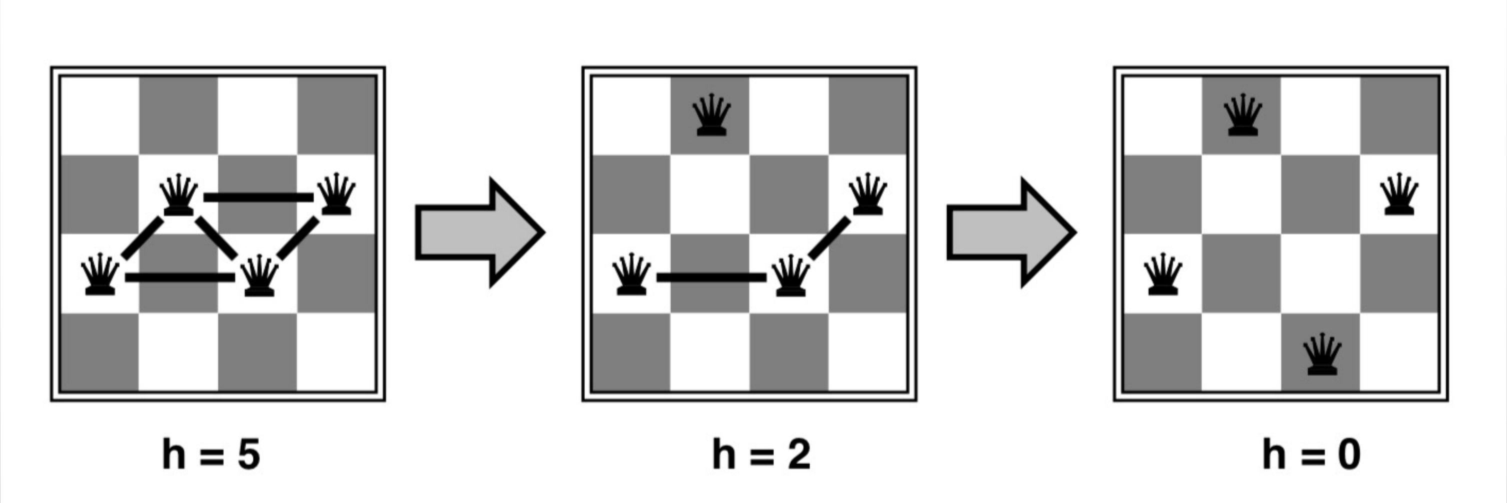
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:

  choose value that violates the fewest constraints

  i.e., hill-climb with h(n) = total number of violated constraints

Example: 4-Queens

- **States**: 4 queens in 4 columns ($4^4 = 256$ states)
- **Operators**: move queen in column
- **Goal test**: no attacks
- **Evaluation**: h(n) = number of attacks



Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure

Summary (cont.)

- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The CSP representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice